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Non-Abelian Chern-Simons vortices with generic gauge groups

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Abstract

We study non-Abelian Chern-Simon BPS-saturated vortices enjoying $\mathcal{N} = 2$ supersymmetry in $d = 2 + 1$ dimensions, with generic gauge groups of the form $U(1) \times G'$, with G' being a simple group, allowing for orientational modes in the solutions. We will keep the group as general as possible and utilizing the powerful moduli matrix formalism to provide the moduli spaces of vortices and derive the corresponding master equations. Furthermore, we study numerically the vortices applying a radial Ansatz to solve the obtained master equations and we find especially a splitting of the magnetic fields, when the coupling constants for the trace-part and the traceless part of the Chern-Simons term are varied, such that the Abelian magnetic field density can become negative near the origin of the vortex while the non-Abelian part stays positive, and vice versa.

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1 Introduction

Solitons play a crucial role in a vast area of physics ranging from particle physics and cosmology to condensed matter physics. Planar physics i.e. in 2+1 dimensions, has radically different properties as the spin is not quantized as in 3+1 dimensions giving rise to the anyons among others, objects having fractional spin and statistics. This can be realized by the Chern-Simons term which has been widely used in e.g. the theory of the fractional quantum Hall effect [1]. Another aspect is that the high temperature limit of a four dimensional theory can be described by a three dimensional one, where the Chern-Simons term resides naturally. Another interesting feature of Chern-Simons theories is that it provides a gauge invariant mechanism of mass generation [2].

The most celebrated vortex solution, namely the Abrikosov-Nielsen-Olesen (ANO) vortex was found half a century ago [3, 4]. This object carries magnetic flux in its interior. Later, similar vortex solutions, however in 2+1 dimensions where found with a Chern-Simons term instead of a Maxwell term [5, 6]. These vortices possess the already mentioned features of fractional spin and statistics, viz. they are anyon-like. Furthermore, there exist vortices in both the asymmetric phase (like for the ANO vortices) and also in the symmetric phase. The latter do not have a topological argument for stability. The vortices with Maxwell or with Chern-Simons terms split into three categories depending on the self-coupling of the Higgs field, viz. type I/II vortices or the critical BPS saturated vortices [7], where the vortices attract, repel and do not feel any force among the selves, respectively. The latter corresponds to some amount of supersymmetry being present in the theories at hand. Recently, a fourth type of vortices in the Abelian Chern-Simons model has been found, behaving as a type I vortex at small amount of magnetic field and turns into type II when the magnetic field piles up repelling further vortices from the clusters [8]. This type of vortex was denoted a type III vortex.

A few years ago, non-Abelian vortices have been discovered [9, 10], being flux tubes which are carrying orientational modes. These models have been extensively studied with the gauge group $U(N) \simeq U(1) \times SU(N)/\mathbb{Z}_N$ and only recently with generalizations to other groups [11, 12, 13]. In particular the moduli space of these vortices have been studied in detail [9, 10, 13, 14, 15, 16]. Good reviews summarizing many results can be found in Refs. [17, 18, 19].

The first studies of non-Abelian Chern-Simons vortices are made with a simple group, viz. $SU(2)$ and $SU(N)$ with fields in the adjoint representation [20, 21, 22] and later numerical solutions have been found [23]. In Refs. [24, 25] the non-Abelian Chern-Simons vortices have been studied with a $U(N)$ gauge group allowing for orientational modes to be present and they identified the moduli space of a single vortex solution. Furthermore Refs. [26, 27, 28] have considered packag-

ing together the Yang-Mills and the non-Abelian Chern-Simons terms for $U(N)$ gauge groups. In Ref. [26] the dynamics of the vortices has been studied and in Ref. [27] in addition to the topological charge, conserved Noether charges associated with a $U(1)^{N-1}$ flavor symmetry of the theory due to inclusion of a mass term for the squarks. In Ref. [28] numerical solutions have been provided.

Many related topics can be found in the excellent reviews [29, 30].

It is the purpose of this paper to consider a wider class of non-Abelian Chern-Simons vortices carrying orientational modes, with the gauge group kept as general as possible, except when we will do some concrete numerical calculations.

2 The model

Our starting point will be the Yang-Mills-Chern-Simons-Higgs theory. We are considering the following $\mathcal{N}=2$ supersymmetric theory (viz. with 4 supercharges) in $d=2+1$ dimensions with the gauge group $G=U(1)\times G'$, where G' is a simple group. The bosonic part of the Lagrangian density reads

$$\begin{aligned}\mathcal{L}_{\rm YMCSH} =& -\frac{1}{4g^2}\left(F^a_{\mu\nu}\right)^2 - \frac{1}{4e^2}\left(F^0_{\mu\nu}\right)^2 - \frac{\mu}{8\pi}\epsilon^{\mu\nu\rho}\left(A^a_\mu\partial_\nu A^a_\rho - \frac{1}{3}f^{abc}A^a_\mu A^b_\nu A^c_\rho\right) - \frac{\kappa}{8\pi}\epsilon^{\mu\nu\rho}A^0_\mu\partial_\nu A^0_\rho \\ &+ \frac{1}{2g^2}\left(\mathcal{D}_\mu\phi^a\right)^2 + \frac{1}{2e^2}\left(\partial_\mu\phi^0\right)^2 + {\rm Tr}\left(\mathcal{D}_\mu H\right)\left(\mathcal{D}^\mu H\right)^\dagger - {\rm Tr}\left|\phi H - Hm\right|^2 \\ &- \frac{g^2}{2}\left({\rm Tr}\left(HH^\dagger t^a\right) - \frac{\mu}{4\pi}\phi^a\right)^2 - \frac{e^2}{2}\left({\rm Tr}\left(HH^\dagger t^0\right) - \frac{\kappa}{4\pi}\phi^0 - \frac{1}{\sqrt{2N}}\xi\right)^2 \ , \end{aligned} \tag{2.1}$$

where $a=1,\ldots,\dim(G')$, the index 0 is for the Abelian group and $\alpha=0,1,\ldots,\dim(G')$ and we use the conventions

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\left[A_\mu, A_\nu\right] \ , \quad \mathcal{D}_\mu H = \left(\partial_\mu + iA_\mu\right)H \ , \quad \mathcal{D}_\mu\phi = \partial_\mu\phi + i\left[A_\mu,\phi\right] \ . \tag{2.2}$$

$A_\mu = A^\alpha_\mu t^\alpha$ is the gauge potential, $F_{\mu\nu}$ is the field strength, ϕ is an adjoint scalar field which we can take to be real and finally H is a color-flavor matrix of dimension $N\times N_{\rm f}$ of $N_{\rm f}$ matter fields. We will define $N\equiv\dim(R_{G'})$ but for simplicity we choose the representation $R_{G'}$ as the fundamental one of G'. We are using the following normalization of the generators

$$t^0 = \frac{\mathbf{1}_N}{\sqrt{2N}} \ , \quad {\rm Tr}\left(t^a t^b\right) = \frac{1}{2}\delta^{ab} \ . \tag{2.3}$$

There are four coupling constants entering our game at this point; $e\in\mathbb{R}$ is the Abelian coupling of the Yang-Mills kinetic term (Maxwell), $g\in\mathbb{R}$ the is the coupling for the semi-simple part

of the Yang-Mills kinetic term, which corresponds to G'. $\kappa \in \mathbb{R}$ is the Abelian coupling of the Chern-Simons term while $\mu \in \mathbb{Z}$ are solely integers to render the non-Abelian Chern-Simons action gauge invariant up to large gauge transformations [31]. ξ is a Fayet-Iliopoulos parameter. Finally, m is a mass matrix which we will set to zero in this paper.

The scope of study in this paper will be on the Chern-Simons part of this theory. A detailed study of the vortices dependence of the parameters of the model above with also the Yang-Mills term in action will be done elsewhere [32].

3 Non–Abelian Chern–Simons–Higgs theory

Now let us take the limit $e \to \infty, g \to \infty, m = 0$ and $\kappa \neq \mu$ and in turn integrate out the adjoint scalar field ϕ:

$$\phi^a = \frac{4\pi}{\mu} \mathrm{Tr}\left(HH^\dagger t^a\right) \ , \quad \phi^0 = \frac{4\pi}{\kappa}\frac{1}{\sqrt{2N}}\left[\mathrm{Tr}\left(HH^\dagger\right) - \xi\right] \ . \tag{3.1}$$

This leaves us with the non-Abelian Chern-Simons theory

$$\begin{aligned}\mathcal{L}_{\mathrm{CSH}} = &-\frac{\mu}{8\pi}\epsilon^{\mu\nu\rho}\left(A^a_\mu \partial_\nu A^a_\rho - \frac{1}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho\right) - \frac{\kappa}{8\pi}\epsilon^{\mu\nu\rho}\left(A^0_\mu \partial_\nu A^0_\rho\right) + \mathrm{Tr}\left(\mathcal{D}_\mu H\right)^\dagger\left(\mathcal{D}^\mu H\right) \\ &- 4\pi^2 \mathrm{Tr}\left|\left\{\frac{\mathbf{1}_N}{N\kappa}\left(\mathrm{Tr}\left(HH^\dagger\right) - \xi\right) + \frac{2}{\mu}\mathrm{Tr}\left(HH^\dagger t^a\right) t^a\right\} H\right|^2 \ ,\end{aligned} \tag{3.2}$$

which will be the main focus of this paper. It still enjoys $\mathcal{N} = 2$ supersymmetry and there are 3 parameters governing the solutions; the Abelian Chern-Simons coupling κ and the non-Abelian Chern-Simons coupling μ and finally the winding number $\nu = \frac{k}{n_0}$ [12]. n_0 denotes the greatest common divisor (gcd) of the Abelian charges of the holomorphic invariants of G', see [12]. For simple groups this coincides with the center as $\mathbb{Z}_{n_0}$. We will take $k > 0$.

There are three different phases of the theory at hand. An unbroken phase with $\langle H \rangle = 0$ and a broken phase with $\langle H \rangle = \sqrt{\frac{\xi}{N}}$. In between there are partially broken phases. We will consider only the completely broken phase in this paper.

The equations of motion are

$$\frac{\mu}{8\pi}\epsilon^{\mu\nu\sigma} F^a_{\mu\nu} = -i \mathrm{Tr}\left[H^\dagger t^a \mathcal{D}^\sigma H - \left(\mathcal{D}^\sigma H\right)^\dagger t^a H\right] \ , \tag{3.3}$$

$$\frac{\kappa}{8\pi}\epsilon^{\mu\nu\sigma} F^0_{\mu\nu} = -i \mathrm{Tr}\left[H^\dagger t^0 \mathcal{D}^\sigma H - \left(\mathcal{D}^\sigma H\right)^\dagger t^0 H\right] \ , \tag{3.4}$$

$$\begin{aligned}\mathcal{D}_\mu\mathcal{D}^\mu H = &-4\pi^2\left[\frac{\mathbf{1}_N}{N\kappa}\left(\mathrm{Tr}\left(HH^\dagger\right)-\xi\right)+\frac{2}{\mu}\mathrm{Tr}\left(HH^\dagger t^a\right)t^a\right]^2 H \\ &-\frac{8\pi^2}{N\kappa}\mathrm{Tr}\left(\left[\frac{\mathbf{1}_N}{N\kappa}\left(\mathrm{Tr}\left(HH^\dagger\right)-\xi\right)+\frac{2}{\mu}\mathrm{Tr}\left(HH^\dagger t^a\right)t^a\right]HH^\dagger\right)H \\ &-\frac{16\pi^2}{\mu}\mathrm{Tr}\left(\left[\frac{\mathbf{1}_N}{N\kappa}\left(\mathrm{Tr}\left(HH^\dagger\right)-\xi\right)+\frac{2}{\mu}\mathrm{Tr}\left(HH^\dagger t^b\right)t^b\right]HH^\dagger t^a\right)t^a H\ . \end{aligned} \tag{3.5}$$

The tension, defined by the integral on the plane over the time-time component of the energy-momentum tensor, is given by

$$T = \int_{\mathbb{C}} \mathrm{Tr}\left\{|\mathcal{D}_0 H|^2+|\mathcal{D}_i H|^2+4\pi^2\left|\left(\frac{\mathbf{1}_N}{N\kappa}\left(\mathrm{Tr}\left(HH^\dagger\right)-\xi\right)+\frac{2}{\mu}\mathrm{Tr}\left(HH^\dagger t^a\right)t^a\right)H\right|^2\right\}\ , \tag{3.6}$$

which by a standard Bogomol'nyi completion can be rewritten as

$$\begin{aligned} T = &\int_{\mathbb{C}}\mathrm{Tr}\left\{\left|\mathcal{D}_0 H - i2\pi\left(\frac{\mathbf{1}_N}{N\kappa}\left(\mathrm{Tr}\left(HH^\dagger\right)-\xi\right)+\frac{2}{\mu}\mathrm{Tr}\left(HH^\dagger t^a\right)t^a\right)H\right|^2+4\left|\bar{\mathcal{D}}H\right|^2\right\} \\ &-\frac{\xi}{\sqrt{2N}}\int_{\mathbb{C}}F_{12}^0 + i\mathrm{Tr}\int_{\mathbb{C}}\left[\partial_2\left(H^\dagger\mathcal{D}_1 H\right)-\partial_1\left(H^\dagger\mathcal{D}_2 H\right)\right]\ . \end{aligned} \tag{3.7}$$

This leads immediately to the BPS-equations which need to be accompanied by the Gauss law being the $\sigma = 0$ component of the Eqs. (3.3),(3.4)

$$\bar{\mathcal{D}}H = 0\ ,\quad \mathcal{D}_0 H = i2\pi\left(\frac{\mathbf{1}_N}{N\kappa}\left(\mathrm{Tr}\left(HH^\dagger\right)-\xi\right)+\frac{2}{\mu}\mathrm{Tr}\left(HH^\dagger t^a\right)t^a\right)H\ . \tag{3.8}$$

Rewriting the boundary term using the first BPS-equation, we have for the BPS saturated vortices the tension

$$T = -\frac{\xi}{\sqrt{2N}}\int_{\mathbb{C}}F_{12}^0+\frac{1}{2}\mathrm{Tr}\int_{\mathbb{C}}\partial_i^2\left(HH^\dagger\right) = 2\pi\xi\nu\ , \tag{3.9}$$

with ν being the $U(1)$ winding number. By combining the BPS equations with the Gauss law, we obtain the following system

$$\bar{\mathcal{D}}H = 0\ , \tag{3.10}$$

$$F_{12}^a t^a = \frac{16\pi^2}{N\kappa\mu}\left(\mathrm{Tr}\left(HH^\dagger\right)-\xi\right)\mathrm{Tr}\left(HH^\dagger t^a\right)t^a+\frac{16\pi^2}{\mu^2}\mathrm{Tr}\left(HH^\dagger t^b\right)\mathrm{Tr}\left(HH^\dagger\left\{t^a,t^b\right\}\right)t^a\ , \tag{3.11}$$

$$F_{12}^0 t^0 = \frac{8\pi^2}{N^2\kappa^2}\mathrm{Tr}\left(HH^\dagger\right)\left(\mathrm{Tr}\left(HH^\dagger\right)-\xi\right)\mathbf{1}_N+\frac{16\pi^2}{N\kappa\mu}\left(\mathrm{Tr}\left(HH^\dagger t^a\right)\right)^2\mathbf{1}_N\ . \tag{3.12}$$

An interesting comment is that the system only depends on three combinations of the couplings; viz. κ^2, μ^2 and $\kappa\mu$. There are thus only two choices of signs giving different solutions sign$(\kappa) = \pm$sign(μ). This system is of a generic character and one can readily apply one's favorite group. Setting $\kappa = \mu$, the BPS-equations become

$$\bar{\mathcal{D}}H = 0\ ,\quad \mathcal{D}_0 H = \frac{i2\pi}{\kappa}\left[2\mathrm{Tr}\left(HH^\dagger t^\alpha\right)t^\alpha-\frac{\xi}{N}\mathbf{1}_N\right]H\ , \tag{3.13}$$

which in turn yields the simplified system by combination with the Gauss law

$$\bar{\mathcal{D}}H = 0 \ , \quad F_{12}^{\alpha} t^{\alpha} = \frac{16\pi^2}{\kappa^2} \left[\mathrm{Tr}\left(HH^\dagger \left\{ t^\alpha, t^\beta \right\} \right) \mathrm{Tr}\left(HH^\dagger t^\beta \right) - \frac{\xi}{N} \mathrm{Tr}\left(HH^\dagger t^\alpha \right) \right] t^\alpha \ . \tag{3.14}$$

In the next section, we will consider the cases of $G' = SU(N)$, $G' = SO(N)$ and $G' = USp(2M)$, and finally make the corresponding master equations.

3.1 Master equations

3.1.1 $G' = U(1) \times SU(N)$

Considering the case of $U(1) \times SU(N)$, the BPS-equations combined with the Gauss law read

$$\begin{aligned}
&\bar{\mathcal{D}}H = 0 \ , \qquad\qquad (3.15)\\
&F_{12}^{a} t^{a} = \frac{8\pi^2}{N\kappa\mu} \left(\mathrm{Tr}\left(HH^\dagger \right) - \xi \right) \left(HH^\dagger - \frac{\mathbf{1}_N}{N} \mathrm{Tr}\left(HH^\dagger \right) \right) \\
&\qquad + \frac{8\pi^2}{\mu^2} \left[HH^\dagger \left(HH^\dagger - \frac{\mathbf{1}_N}{N} \mathrm{Tr}\left(HH^\dagger \right) \right) - \frac{\mathbf{1}_N}{N} \mathrm{Tr}\left(\left(HH^\dagger \right)^2 \right) + \frac{\mathbf{1}_N}{N^2} \left(\mathrm{Tr}\left(HH^\dagger \right) \right)^2 \right] \ , \\
&F_{12}^{0} t^{0} = \frac{8\pi^2}{N^2\kappa^2} \mathrm{Tr}\left(HH^\dagger \right) \left(\mathrm{Tr}\left(HH^\dagger \right) - \xi \right) \mathbf{1}_N + \frac{8\pi^2}{N\kappa\mu} \left[\mathrm{Tr}\left(\left(HH^\dagger \right)^2 \right) - \frac{1}{N} \left(\mathrm{Tr}\left(HH^\dagger \right) \right)^2 \right] \mathbf{1}_N \ .
\end{aligned}$$

In this case, the generic vacuum is given by

$$\langle H \rangle = \sqrt{\frac{\xi}{N}} \mathbf{1}_N \ . \tag{3.16}$$

This vacuum allows for an unbroken global symmetry, the so-called color-flavor symmetry which is the global part of the gauge transformation combined with the flavor symmetry. This is of crucial importance for having orientational modes in vortex configurations.

Utilizing the moduli matrix formalism, we can immediately solve the first BPS-equation and rewrite the second in terms of the new variables

$$H = S^{-1} H_0(z) \ , \quad \bar{A}^a t^a = -i S'^{-1} \bar{\partial} S' \ , \quad \bar{A}^0 t^0 = -i \bar{\partial} \log s \tag{3.17}$$

along with the definitions $\Omega \equiv \omega \Omega', \Omega' \equiv S' S'^\dagger, \omega \equiv s s^\dagger$ and $\Omega_0 \equiv H_0(z) H_0^\dagger(z)$. The field-strength matrices are

$$F_{12}^{a} t^{a} = 2 S'^{-1} \bar{\partial} \left[\Omega' \partial \Omega'^{-1} \right] S' \ , \qquad F_{12}^{0} t^{0} = -2 \mathbf{1}_N \bar{\partial} \partial \log \omega \ . \tag{3.18}$$

In this $U(1)\times SU(N)$ case we can write down the two master equations like

$$\begin{aligned}\bar\partial\left[\Omega'\partial\Omega'^{-1}\right] &= \frac{4\pi^2}{N\kappa\mu}\frac{1}{\omega}\left(\frac{1}{\omega}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)-\xi\right)\left(\Omega_0\Omega'^{-1}-\frac{\mathbf{1}_N}{N}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)\right)\\ &\quad+\frac{4\pi^2}{\mu^2}\frac{1}{\omega^2}\Bigg[\Omega_0\Omega'^{-1}\left(\Omega_0\Omega'^{-1}-\frac{\mathbf{1}_N}{N}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)\right)\\ &\qquad-\frac{\mathbf{1}_N}{N}\mathrm{Tr}\left(\left(\Omega_0\Omega'^{-1}\right)^2\right)+\frac{\mathbf{1}_N}{N^2}\left(\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)\right)^2\Bigg]\ , \end{aligned}\tag{3.19}$$

$$\begin{aligned}\bar\partial\partial\log\omega &= -\frac{4\pi^2}{N^2\kappa^2}\frac{1}{\omega}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)\left(\frac{1}{\omega}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)-\xi\right)\\ &\quad+\frac{4\pi^2}{N\kappa\mu}\frac{1}{\omega^2}\left[\mathrm{Tr}\left(\left(\Omega_0\Omega'^{-1}\right)^2\right)-\frac{1}{N}\left(\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)\right)^2\right]\ .\end{aligned}\tag{3.20}$$

Setting the couplings equal $\kappa=\mu$, we can write the $U(N)$ Chern-Simons BPS-equations and master equation as simple as

$$F_{12}^{\alpha}t^{\alpha} = \frac{8\pi^2}{\kappa^2}HH^\dagger\left(HH^\dagger-\frac{\xi}{N}\mathbf{1}_N\right)\ ,\tag{3.21}$$

$$\bar\partial\left[\Omega\partial\Omega^{-1}\right] = \frac{4\pi^2}{\kappa^2}\Omega_0\Omega^{-1}\left[\Omega_0\Omega^{-1}-\frac{\xi}{N}\mathbf{1}_N\right]\ .\tag{3.22}$$

The boundary conditions for these master equations coincide with the weak coupling solutions (3.47).

3.1.2 $G'=U(1)\times SO(N)$ and $G'=U(1)\times USp(2M)$

Considering now the gauge group $G=U(1)\times SO(N)$ and $G=U(1)\times USp(2M)$ on the same footing with their corresponding invariant tensor J, which has the properties $J^\dagger J=\mathbf{1}_N$ and $J^{\mathrm{T}}=\epsilon J$ with $\epsilon=\pm1$ for $SO(N)$ and $USp(2M)$, respectively.

The vacuum has the generic form [33]

$$\langle H\rangle = \mathrm{diag}\left(v_1,v_2,\ldots,v_N\right)\ ,\quad v_i\in\mathbb{R}_+\ ,\tag{3.23}$$

however, we will consider the most symmetric vacuum allowing for the global color-flavor symmetry, viz. we will here use (3.16). We have the following system which is obtained by combining the BPS equations with the Gauss law and applying respective algebras

$$\bar{\mathcal{D}}H = 0\ ,\tag{3.24}$$

$$F_{12}^a t^a = \frac{4\pi^2}{N\kappa\mu}\left(\mathrm{Tr}\left(HH^\dagger\right)-\xi\right)\left(HH^\dagger-J^\dagger\left(HH^\dagger\right)^{\mathrm{T}}J\right)+\frac{2\pi^2}{\mu^2}\left[\left(HH^\dagger\right)^2-J^\dagger\left(\left(HH^\dagger\right)^2\right)^{\mathrm{T}}J\right]\ ,$$

$$F_{12}^0t^0 = \frac{8\pi^2}{N^2\kappa^2}\mathrm{Tr}\left(HH^\dagger\right)\left(\mathrm{Tr}\left(HH^\dagger\right)-\xi\right)\mathbf{1}_N+\frac{4\pi^2}{N\kappa\mu}\mathrm{Tr}\left(HH^\dagger\left(HH^\dagger-J^\dagger\left(HH^\dagger\right)^{\mathrm{T}}J\right)\right)\mathbf{1}_N\ ,$$

which lead to the master equations

$$\bar{\partial}\left[\Omega'\partial\Omega'^{-1}\right] = \frac{2\pi^2}{N\kappa\mu}\frac{1}{\omega}\left(\frac{1}{\omega}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)-\xi\right)\left(\Omega_0\Omega'^{-1}-J^\dagger\left(\Omega_0\Omega'^{-1}\right)^{\mathrm{T}}J\right)$$
$$+\frac{\pi^2}{\mu^2}\frac{1}{\omega^2}\left[\left(\Omega_0\Omega'^{-1}\right)^2-J^\dagger\left(\left(\Omega_0\Omega'^{-1}\right)^2\right)^{\mathrm{T}}J\right] , \tag{3.25}$$

$$\bar{\partial}\partial\log\omega = -\frac{4\pi^2}{N^2\kappa^2}\frac{1}{\omega}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)\left(\frac{1}{\omega}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)-\xi\right)$$
$$-\frac{2\pi^2}{N\kappa\mu}\frac{1}{\omega^2}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\left(\Omega_0\Omega'^{-1}-J^\dagger\left(\Omega_0\Omega'^{-1}\right)^{\mathrm{T}}J\right)\right) . \tag{3.26}$$

The boundary conditions for these master equations coincide with the weak coupling solutions (3.51).

3.1.3 Energy density and flux densities

Rewriting the energy density (3.9) in terms of our new variables and remembering the boundary term which vanishes when integrating over the entire plane, while nevertheless produces a big difference between the magnetic flux density and the energy density, we have

$$\mathcal{E} = 2\xi\bar{\partial}\partial\log\omega + 2\bar{\partial}\partial\left(\frac{1}{\omega}\mathrm{Tr}\Omega_0\Omega'^{-1}\right) . \tag{3.27}$$

However, the total energy

$$E = \int_{\mathbb{C}}\mathcal{E} = 2\pi\xi\nu = \frac{2\pi\xi k}{n_0} , \tag{3.28}$$

is simply proportional to the topological charge as always.

The Abelian magnetic flux density is the first term (up to a factor) in the energy density

$$\mathcal{B} = F_{12}^0 = -2\sqrt{2N}\;\bar{\partial}\partial\log\omega , \tag{3.29}$$

whereas the non-Abelian flux is the matrix defined in Eq. (3.18). The Abelian electric field density reads

$$E_i = F_{i0}^0 = \frac{2\pi}{\kappa}\sqrt{\frac{2}{N}}\partial_i\left(\frac{1}{\omega}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)\right) , \tag{3.30}$$

while the non-Abelian electric field density is given by

$$E_i^a t^a = F_{i0}^a t^a = \frac{4\pi}{\mu}\partial_i\mathrm{Tr}\left(HH^\dagger t^a\right)t^a . \tag{3.31}$$

This can be written for $G' = SU(N)$ as

$$E_i^a t^a = \frac{2\pi}{\mu}\partial_i\left[\frac{1}{\omega}\left(S'^{-1}\Omega_0\Omega'^{-1}S' - \frac{1}{N}\mathrm{Tr}\left(\Omega_0\Omega'^{-1}\right)\right)\right] , \tag{3.32}$$

while for $G' = SO(N)$ or $G' = USp(2M)$ it is

$$E_i^a t^a = \frac{\pi}{\mu}\partial_i \left[\frac{1}{\omega} S'^{-1}\left(\Omega_0 \Omega'^{-1} - J^\dagger \left(\Omega_0\Omega'^{-1}\right)^{\mathrm{T}} J\right) S'\right] . \tag{3.33}$$

3.2 Solutions

In the Abelian Chern-Simons theory, there exists a rigorous existence proof of the solutions in Ref. [34]. To our knowledge this has not rigorously been proved in the theory at hand. In the case of the vortices in the Yang-Mills-Higgs theory, the "covariant holomorphic" condition on the Higgs fields $\bar{D}H = 0$, which is solved by the moduli matrix formalism, does uniquely determine the full moduli space of vortices via the Hitchin-Kobayashi correspondence [35, 36, 37, 38], which however has only been proved on compact spaces. This means that the corresponding master equations do not induce further moduli. For the vortices with the $U(N)$ gauge group, an index theorem has been given in Ref. [9] while for generic gauge groups (under certain conditions) an index theorem has been given in Ref. [13]. The index computed gives the number of moduli and does indeed correspond to the number of moduli found in the moduli matrix.

The first part of constructing a solution is to write down the moduli matrix. Here we simply follow the way paved by the paper [12] using holomorphic invariants of the gauge subgroup G'. This boils down to some constraints for the moduli matrix to obey. A few examples of interest here is the case of $G' = SU(N)$

$$\det H_0(z) = z^k + \mathcal{O}\left(z^{k-1}\right) , \tag{3.34}$$

while in the case of $G' = SO, USp$, respectively, we have

$$H_0^{\mathrm{T}}(z) J H_0(z) = z^{\frac{2k}{n_0}} J + \mathcal{O}\left(z^{\frac{2k}{n_0}-1}\right) , \tag{3.35}$$

where k is the vortex number (recall that $\nu = \frac{k}{n_0}$ is the $U(1)$ winding) and $n_0 = 2$ in case of $SO(2M)$ and $USp(2M)$ while $n_0 = 1$ for $SO(2M+1)$, M being positive integers. For $SU(N)$, however $n_0 = N$.

The rather complicated looking master equations found in the last section are assumed to have a unique solution for each moduli matrix $H_0(z)$ (up to V equivalence, see Ref. [12]). That is the moduli matrices are redundant and have to be identified by the following V transformation

$$H_0(z) \sim V(z,\bar{z}) H_0(z) , \quad S(z,\bar{z}) \sim V(z,\bar{z}) S(z,\bar{z}) , \quad V \in G^{\mathbb{C}} . \tag{3.36}$$

Here we conjecture the existence and uniqueness of the solutions to the master equation for each moduli matrix (up to the V equivalence). To provide plausibility for this claim we shall continue in two directions.

First we consider the weak coupling limit $\kappa \to 0$ and $\mu \to 0$, which seems like an odd limit to take, but having an advantage. Looking at the theory (2.1) it is immediately seen that the matter fields are forced to stay in the vacuum manifold corresponding to the *strong* coupling limit of the normal non-Abelian vortex (i.e. with only a Yang-Mills kinetic term). In turn, this gives us a unique solution which in fact is the same solution as found in the strong coupling limit of the non-Abelian vortex with only a Yang-Mills kinetic term. This solution, appropriate only for vortices of the semi-local type, are usually called lumps in the literature.

The second direction we will take will simply be to find some solutions by numerical calculations.

Now the existence of the solutions to the master equations, as we argue, makes it possible to exploit a lot of results developed in the literature. In short,

the moduli space of non-Abelian Chern-Simons k vortices with gauge group G
is equal to the moduli space of the non-Abelian Yang-Mills k vortices with gauge group G.

(3.37)

Moduli spaces of the non-Abelian vortices in $\mathcal{N}=2$ sQCD has been found in the literature in Refs. [9, 14] for $U(N)$ and in Refs. [13] for $SO(N), USp(2M)$.

Here we will summarize a few results from the literature. In the pioneering papers [9, 10] discovering the non-Abelian vortices with gauge group $U(N)$ (in contrast to the formerly found $\mathbb{Z}_N$ strings) the moduli space of a single vortex string was found to be

$$\mathcal{M}_{k=1,G'=SU(N)} = \mathbb{C} \times \mathbb{C}P^{N-1}, \tag{3.38}$$

where the first factor denotes the position in the transverse plane while the second factor are orientational modes. For well separated k vortices, the moduli space can be composed as simply the symmetric product of that of the single vortex. This is not the case, when the centers coincide. In the $k=2$, $U(2)$ case, the moduli space has been found explicitly in the Refs. [15, 16]

$$\mathcal{M}_{k=2,G'=U(2)} = \mathbb{C} \times W\mathbb{C}P^2_{2,1,1}\ , \tag{3.39}$$

which decomposes into a center-of-mass position and a weighted complex projective space with unequal weights giving rise to a conical type of singularity. In Ref. [13] the moduli spaces of vortices with gauge groups $G = U(1) \times SO(N)$ and $G = U(1) \times USp(2M)$ has been found. A complication arises due to the fact that already for $N_{\rm f} = N$ flavors, the vortices are in general of the semi-local type (i.e. they have polynomial tails in their profile functions). The spaces quoted here correspond to the vortices of local type, thus they are constrained to have holomorphic

invariants with coincident zeroes. In the language of Ref. [13] this is obtained by constraining the vortices by the so-called strong condition

$$H_0^{\mathrm{T}}(z) J H_0(z) = (z - z_0)^{\frac{2k}{n_0}} J \ . \tag{3.40}$$

The single local vortex with $G' = USp(2M)$ has the moduli space

$$\mathcal{M}_{k=1,G'=USp(2M)} = \mathbb{C} \times \frac{USp(2M)}{U(M)} \ , \tag{3.41}$$

while in the case of $G' = SO(2M)$ it is found to be

$$\mathcal{M}_{k=1,G'=SO(2M)} = \left(\mathbb{C} \times \frac{SO(2M)}{U(M)}\right)_+ \cup \left(\mathbb{C} \times \frac{SO(2M)}{U(M)}\right)_- \ , \tag{3.42}$$

where the $\pm$ denotes the chirality as described in detail in Ref. [13] which is deeply rooted in the fact that the first homotopy group has in addition to the integers a $\mathbb{Z}_2$ factor. This can also be interpreted as two spinor representations which is exactly the irreducible representations of the dual group $\tilde{G}'$, where the dual is defined as being the group having the root vectors $\vec{\alpha}^* = \frac{\vec{\alpha}}{\vec{\alpha}\cdot\vec{\alpha}}$. For the $k = 2$, $G' = SO(2M)$ the following orientational moduli spaces have been found to be locally

$$\mathcal{M}_{k=2,G'=SO(4m),Q_{\mathbb{Z}_2}=+1} = \mathbb{R}_+^m \times \frac{SO(4m)}{USp(2)^m} \times \mathbb{Z}_2 \ , \tag{3.43}$$

$$\mathcal{M}_{k=2,G'=SO(4m),Q_{\mathbb{Z}_2}=-1} = \mathbb{R}_+^{m-1} \times \frac{SO(4m)}{U(1) \times USp(2)^{m-1} \times SO(2)} \ , \tag{3.44}$$

$$\mathcal{M}_{k=2,G'=SO(4m+2),Q_{\mathbb{Z}_2}=+1} = \mathbb{R}_+^m \times \frac{SO(4m+2)}{U(1) \times USp(2)^m} \times \mathbb{Z}_2 \ , \tag{3.45}$$

$$\mathcal{M}_{k=2,G'=SO(4m+2),Q_{\mathbb{Z}_2}=-1} = \mathbb{R}_+^m \times \frac{SO(4m+2)}{USp(2)^m \times SO(2)} \ . \tag{3.46}$$

In the case of $k = 1$, $G' = SO(2M + 1)$, the moduli spaces are quite similar to the $k = 2$ even case.

3.3 Weak coupling limit

3.3.1 $G' = U(1) \times SU(N)$

Taking $\kappa = \mu \to 0$, we obtain from the D term conditions

$$\Omega' = (\det \Omega_0)^{-\frac{1}{N}} \Omega_0 \ , \quad \omega = \frac{N}{\xi} (\det \Omega_0)^{\frac{1}{N}} \ , \quad \Omega = \frac{N}{\xi} \Omega_0 \ , \tag{3.47}$$

which can be packaged together as a $U(N)$ field Ω. Instead of taking both couplings simultaneously to weak coupling, we can play a game of taking only one of them, keeping the other finite (non-infinitesimal). Taking $\kappa \to 0$ and keeping μ finite we obtain

$$\omega = \frac{1}{\xi}\mathrm{Tr}\Omega_0\Omega'^{-1} \ , \tag{3.48}$$

at the zeroth order in κ while at first order we get the constraint

$$N\mathrm{Tr}\left(\left(\Omega_0\Omega'^{-1}\right)^2\right) = \left(\mathrm{Tr}\,\Omega_0\Omega'^{-1}\right)^2 \ . \tag{3.49}$$

We note that only the Abelian field is determined, however at first order in the coupling constant we obtain a single constraint on the non-Abelian fields. Taking instead $\mu \to 0$ keeping κ finite we have

$$\Omega' = \Lambda\Omega_0 \ , \qquad \text{with } \Lambda \in \text{const.} \ , \tag{3.50}$$

to both zeroth and first order in μ.

3.3.2 $G' = U(1) \times SO(N)$ and $G' = U(1) \times USp(2M)$

Taking $\kappa = \mu \to 0$ we have from the D term conditions [13, 33]

$$\Omega' = H_0(z)\frac{\mathbf{1}_N}{\sqrt{M^\dagger M}}H_0^\dagger(z) \ , \qquad \omega = \frac{1}{\xi}\mathrm{Tr}\sqrt{M^\dagger M} \ , \tag{3.51}$$

where $M = H_0^{\mathrm{T}}(z)JH_0(z)$ is the meson field of the SO, USp theories according to the choice of the gauge group and in turn invariant tensor.

A comment in store is that the Chern-Simons term is simply switched off in this limit and the lumps are *the same* as the ones living in the Yang-Mills theories experiencing infinitely massive gauge bosons. The point here, however, is to argue by continuity the existence and uniqueness of the solutions to the master equations for a given moduli matrix $H_0(z)$ (up to the V-equivalence relation).

3.4 Numerical solutions

3.4.1 Example: $U(N)$

Let us do a warm-up and consider the single $U(N)$ Chern-Simons vortex ($\kappa = \mu$) as has been found in Refs. [25, 24], however doing it in our formalism. Taking a simple moduli matrix

$$H_0(z) = \mathrm{diag}\left(z, \mathbf{1}_{N-1}\right) \ , \tag{3.52}$$

which of course satisfies the constraint (3.34), thus we can use the Ansatz for Ω

$$\Omega = e^{\psi}\mathrm{diag}\left(e^{(N-1)\chi}, e^{-\chi}\mathbf{1}_{N-1}\right) \ , \tag{3.53}$$

leading to the two coupled equations of motion

$$\bar{\partial}\partial\left[\psi + (N-1)\chi\right] = -\frac{4\pi^2}{\kappa^2}\left|z - z_0\right|^2 e^{-\psi-(N-1)\chi}\left(\left|z-z_0\right|^2 e^{-\psi-(N-1)\chi} - \frac{\xi}{N}\right) \ , \tag{3.54}$$

$$\bar{\partial}\partial\left[\psi - \chi\right] = -\frac{4\pi^2}{\kappa^2}e^{-\psi+\chi}\left(e^{-\psi+\chi} - \frac{\xi}{N}\right) \ . \tag{3.55}$$

Notice that the two equations decouple in the sense that there only appear the combinations $\psi+(N-1)\chi$ and $\psi-\chi$. In fact it is easily seen that in this case, the field combination $\psi-\chi$ can be in the vacuum in all $\mathbb{C}$ which trivially solves the second equation. However, the first equation still needs to be solved numerically. The boundary conditions are

$$\psi_\infty = \log\left(\frac{N|z|^{\frac{2}{N}}}{\xi}\right) \ , \qquad \chi_\infty = \log\left(|z|^{\frac{2}{N}}\right) \ . \tag{3.56}$$

The equations become essentially Abelian when the couplings are equal $\kappa = \mu$, as was noted in Ref. [25]. The energy density is given by

$$\mathcal{E} = 2\xi\bar{\partial}\partial\psi + 2\bar{\partial}\partial\left[|z|^2 e^{-\psi-(N-1)\chi} + (N-1)e^{-\psi+\chi}\right] \ , \tag{3.57}$$

where the last term is the boundary term which of course integrates to zero. The Abelian and non-Abelian magnetic flux densities are given by

$$F_{12}^0 = -2\sqrt{2N}\ \bar{\partial}\partial\psi \ , \quad F_{12}^a t^a = -2\sqrt{2N(N-1)}\ \bar{\partial}\partial\chi\, t \ , \tag{3.58}$$

where the following matrix has been defined for convenience

$$t \equiv \frac{1}{\sqrt{2N(N-1)}}\mathrm{diag}\left(N-1, -\mathbf{1}_{N-1}\right) \ , \tag{3.59}$$

which is traceless and has the trace of its square normalized to one half. The Abelian electric field density is

$$E_r = \frac{2\pi}{\kappa}\sqrt{\frac{2}{N}}\partial_r\left[r^2 e^{-\psi-(N-1)\chi} + (N-1)e^{-\psi+\chi}\right] \ , \tag{3.60}$$

while the non-Abelian electric field density is

$$E_r^a t^a = \frac{2\pi}{\mu N}\sqrt{2N(N-1)}\partial_r\left[r^2 e^{-\psi-(N-1)\chi} - e^{-\psi+\chi}\right] t \ . \tag{3.61}$$

We will find in the next subsection, that the numerical solution for this vortex for $N=2$ is up to rescaling of some parameters equivalent to the vortex studied in the next subsection (when $\kappa = \mu$). Thus the concrete graphs are shown only for the vortex solution below.

3.4.2 Example: $U(1) \times SO(2M)$ and $U(1) \times USp(2M)$

Let us take a simple example of a moduli matrix

$$H_0(z) = \mathrm{diag}\left(z\mathbf{1}_M, \mathbf{1}_M\right) \ , \tag{3.62}$$

which surely satisfies the constraint (3.35). We take the Ansatz

$$\Omega' = \mathrm{diag}\left(e^{\chi}\mathbf{1}_M, e^{-\chi}\mathbf{1}_M\right) \ , \quad \omega = e^{\psi} \ , \tag{3.63}$$

where $\det\Omega' = 1$ is manifest. The equations of motion in terms of the new fields are

$$\begin{aligned}\bar{\partial}\partial\chi = & -\frac{\pi^2}{\kappa\mu}\left(|z|^2 e^{-\psi-\chi} + e^{-\psi+\chi} - \frac{\xi}{M}\right)\left(|z|^2 e^{-\psi-\chi} - e^{-\psi+\chi}\right) \\ & -\frac{\pi^2}{\mu^2}\left(\left(|z|^2 e^{-\psi-\chi}\right)^2 - \left(e^{-\psi+\chi}\right)^2\right) \ , \end{aligned} \tag{3.64}$$

$$\begin{aligned}\bar{\partial}\partial\psi = & -\frac{\pi^2}{\kappa^2}\left(|z|^2 e^{-\psi-\chi} + e^{-\psi+\chi}\right)\left(|z|^2 e^{-\psi-\chi} + e^{-\psi+\chi} - \frac{\xi}{M}\right) \\ & -\frac{\pi^2}{\kappa\mu}\left(|z|^2 e^{-\psi-\chi} - e^{-\psi+\chi}\right)^2 \ . \end{aligned} \tag{3.65}$$

It is interesting to note that under rescaling of the FI parameter $\xi \to M\xi$, the above equations of motion are exactly the ones of the $U(1) \times SU(2)$ theory with the Ansatz used in the last section. The boundary conditions are

$$\psi_\infty = \log\left(\frac{2M}{\xi}|z|\right) \ , \quad \chi_\infty = \log\left(|z|\right) \ , \tag{3.66}$$

and the energy density reads

$$\mathcal{E} = 2\xi\bar{\partial}\partial\psi + 2M\bar{\partial}\partial\left[|z|^2 e^{-\psi-\chi} + e^{-\psi+\chi}\right] \ , \tag{3.67}$$

where the first term is proportional to the Abelian magnetic flux density

$$F_{12}^0 = -4\sqrt{M}\ \bar{\partial}\partial\psi \ , \tag{3.68}$$

and the last is the boundary term which integrates to zero, while the non-Abelian magnetic field density reads

$$F_{12}^a t^a \equiv F_{12}^{\mathrm{NA}} t = -4\sqrt{M}\ \bar{\partial}\partial\chi\, t \ , \quad t \equiv \frac{1}{2\sqrt{M}}\mathrm{diag}\left(\mathbf{1}_M, -\mathbf{1}_M\right) \ . \tag{3.69}$$

The Abelian electric field density reads

$$E_r = \frac{2\pi\sqrt{M}}{\kappa}\partial_r\left[r^2 e^{-\psi-\chi} + e^{-\psi+\chi}\right] \ , \tag{3.70}$$

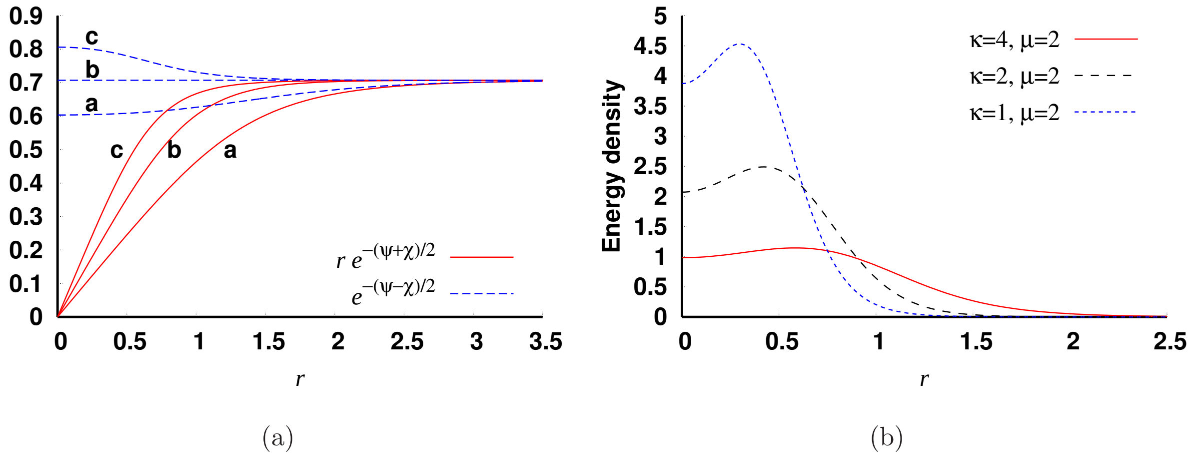


Fig. 1: (a) Profile functions for three different values of the coupling constants; a: $\kappa = 4, \mu = 2$; b: $\kappa = 2, \mu = 2$; c: $\kappa = 1, \mu = 2$; the functions are plotted in traditional style with the winding field rising linearly and the non-winding field being constant at the origin. The FI parameter $\xi = 2$. Notice that the VEV for these functions is $2^{-\frac{1}{2}}$. (b) The energy density $\mathcal{E}$ for the vortex for the same three different values of the couplings. All the energy densities integrate to $\pi\xi$, within an accuracy better than $\sim 10^{-4}$.

whereas the non-Abelian electric field density is

$$E_r^a t^a \equiv E_r^{\rm NA} t = \frac{2\pi\sqrt{M}}{\mu}\partial_r \left[r^2 e^{-\psi-\chi} - e^{-\psi+\chi}\right] t \ . \tag{3.71}$$

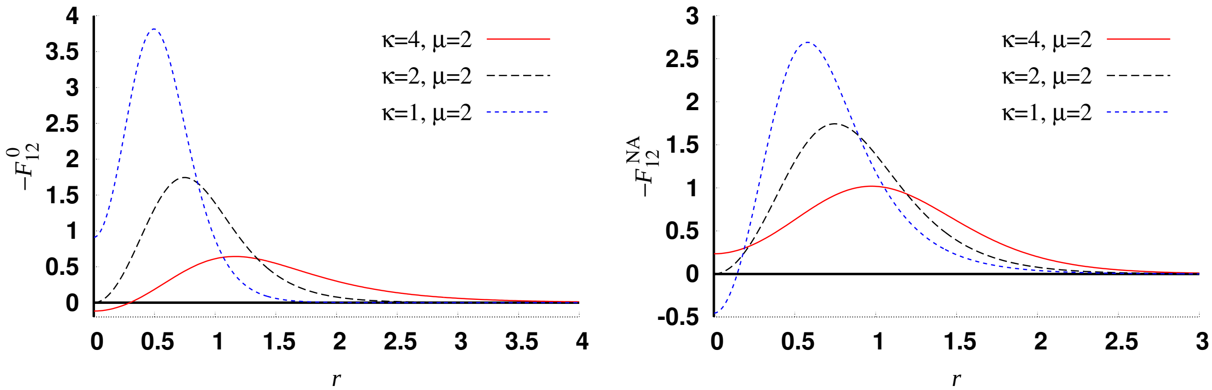
We show the vortex with this Ansatz corresponding to different values of the coupling constants κ, μ in the following figures. Here we will take for definiteness the group G' to be $SO(4)$ or $USp(4)$ hence $M = 2$, which within the chosen Ansatz are equivalent. We furthermore set $\xi = 2$. The total energy is thus (recall the Ansatz is for a single $k = 1$ vortex)

$$E = \int_{\mathbb{C}} \mathcal{E} = \pi\xi \ . \tag{3.72}$$

In Fig. 1a we show the profile functions of the vortex in the traditional way, where the color-flavor matrix is parametrized as follows

$$H = \mathrm{diag}\left(f(r)e^{i\theta}\mathbf{1}_2, g(r)\mathbf{1}_2\right) \ , \tag{3.73}$$

which of course is equivalent to the parametrization in terms of ψ, χ. In Fig. 1b the energy density of Eq. (3.67) is shown. The integral of the energy density is identically equal to the integral of the Abelian magnetic flux, as it should be. We see the vortex size is proportional to the coupling constants. In Fig. 2 we show the Abelian (a) and the non-Abelian (b) magnetic field, respectively. We observe that the Abelian magnetic field is negative at the origin while



(a) (b)

Fig. 2: (a) The Abelian magnetic field F^0_{12} (trace-part) for three different values of the couplings. Notice the equal coupling case has zero magnetic field at the origin while the different coupling cases have negative and positive values, respectively. (b) The non-Abelian magnetic field F^a_{12} (traceless part) for different values of the couplings. Notice the opposite behavior of the non-Abelian magnetic field with respect the Abelian one at the origin, see also Fig. 4. The FI parameter $\xi = 2$.

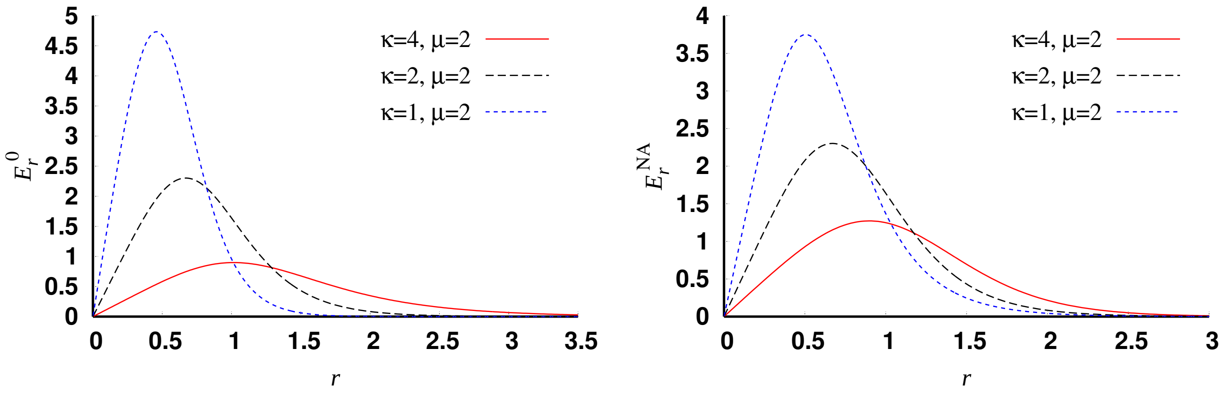
the non-Abelian magnetic field is positive, in the $\kappa = 4, \mu = 2$ case. The contrary holds in the $\kappa = 1, \mu = 2$ case where the non-Abelian magnetic field is negative at the origin while the Abelian field is positive. It turns out that the combination

$$\left(\kappa F^0_{12} + \mu F^{\rm NA}_{12}\right)\Big|_{r\to 0} = 0 \ . \qquad (3.74)$$

An immediate consequence is that for $|\kappa| \gg |\mu|$, $|F^{\rm NA}_{12}| \gg |F^0_{12}|$ at the origin and vice versa. Plots of the Abelian and non-Abelian magnetic fields normalized as in Eq. (3.74) are shown in Fig. 4 with $\kappa = 4, \mu = 2$ in (a) and $\kappa = 1, \mu = 2$ in (b), respectively. At the origin this combination cancels to a numerical accuracy better than 10^{-5}. First let us demonstrate the formula (3.74) by calculating the fields in the limit $r \to 0$

$$\kappa F^0_{12}\Big|_{r\to 0} = -\mu F^{\rm NA}_{12}\Big|_{r\to 0} = 4\pi\sqrt{M}\left[\frac{1}{\kappa}e^{-\psi+\chi}\left(e^{-\psi+\chi} - \frac{\xi}{M}\right) + \frac{1}{\mu}\left(e^{-\psi+\chi}\right)^2\right] \ . \qquad (3.75)$$

Note that the value of the magnetic fields only depends on the field combination $\psi - \chi$, and it is understood that it has to be evaluated at the origin in the above equation. Secondly, let us demonstrate that the magnetic fields are zero at the origin in the case of equal couplings.



(a) (b)

Fig. 3: (a) The Abelian electric field in the radial direction E_r (trace-part) for three different values of the couplings. (b) The non-Abelian electric field in the radial direction $E_r^{\rm NA}$ (traceless part). The FI parameter $\xi = 2$.

Subtracting Eq. (3.64) from Eq. (3.65) we have

$$\bar{\partial}\partial\left(\psi-\chi\right) = -\frac{\pi^2}{\kappa^2}\Bigg[\left(1-\frac{\kappa^2}{\mu^2}\right)\left(|z|^2 e^{-\psi-\chi}\right)^2 + \left(1-\frac{\kappa}{\mu}\right)\left(2e^{-\psi+\chi}-\frac{\xi}{M}\right)|z|^2 e^{-\psi-\chi} \\ + \left(1+\frac{\kappa}{\mu}\right)^2\left(e^{-\psi+\chi}\right)^2 - \frac{\xi}{M}\left(1+\frac{\kappa}{\mu}\right)e^{-\psi+\chi}\Bigg] , \tag{3.76}$$

which depends on $z, \bar{z}$ when the coupling constants are different, $\kappa \neq \mu$. However, when the coupling constants are equal, Eq. (3.76) reads

$$\bar{\partial}\partial\left(\psi-\chi\right) = -\frac{4\pi^2}{\kappa^2}\left(e^{-\psi+\chi}-\frac{\xi}{2M}\right)e^{-\psi+\chi} , \tag{3.77}$$

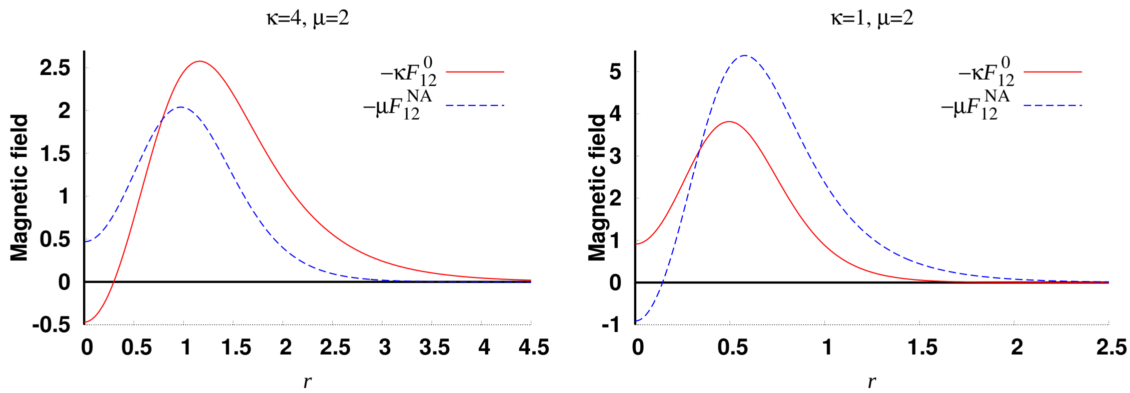
which allows the field combination $\psi-\chi$ to stay constant with the value

$$\psi - \chi = \log\left(\frac{2M}{\xi}\right) . \tag{3.78}$$

Plugging this (constant) solution into Eq. (3.75) we obtain readily $F_{12}^0 = F_{12}^{\rm NA} = 0$ in the limit $r \to 0$.

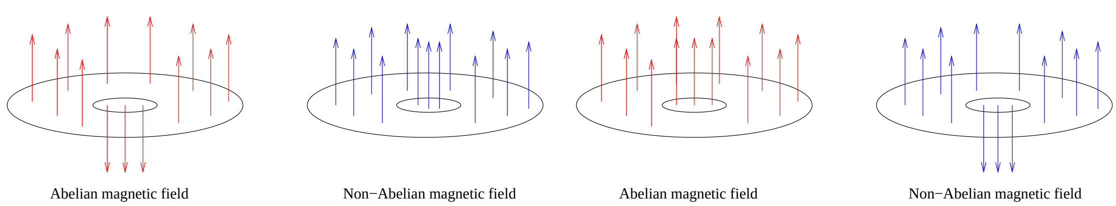
In Fig. 3 is shown the Abelian (a) and non-Abelian (b) electric fields with different values of the couplings.

In Fig. 5 we show a sketch of the magnetic fields of Abelian and non-Abelian kinds, respectively, in the case of $\kappa > \mu$ (a) and in the case of $\kappa < \mu$ (b). The integral over the plane of the Abelian magnetic field density is proportional to the topological charge of the vortex, the winding number which in turn renders the soliton topologically stable. The vortex solution with negative



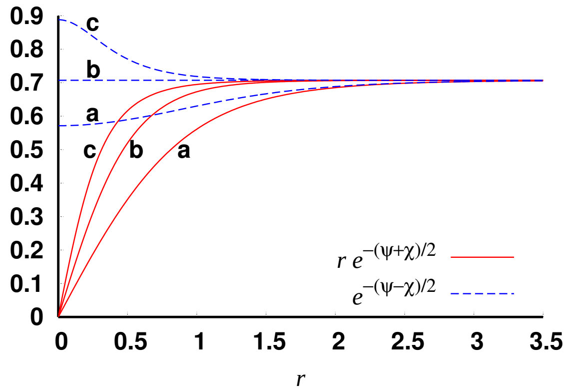
(a) (b)

Fig. 4: Differently normalized Abelian and non-Abelian magnetic fields as κF_{12}^{0} and $\mu F_{12}^{\rm NA}$ for (a) $\kappa = 4, \mu = 2$ and (b) $\kappa = 1, \mu = 2$. This combination cancels exactly at the origin (to a numerical accuracy better than $\sim 10^{-5}$). The FI parameter $\xi = 2$.

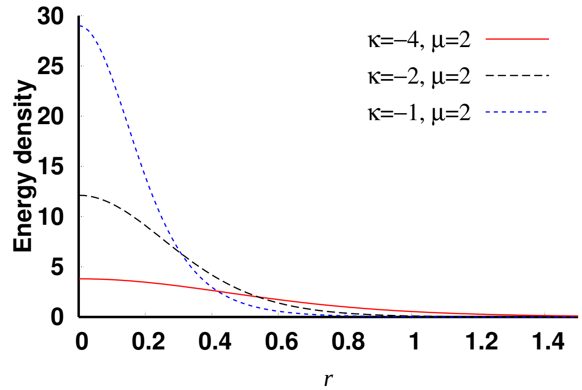


(a) $\kappa > \mu$ (b) $\kappa < \mu$

Fig. 5: (a) Sketch of the magnetic fields where the Abelian (red/left) is negative at the origin and the non-Abelian (blue/right) is positive for $\kappa > \mu$. (b) Contrarily the Abelian (red/left) is positive at the origin while the non-Abelian (blue/right) is negative for $\kappa < \mu$.



(a)



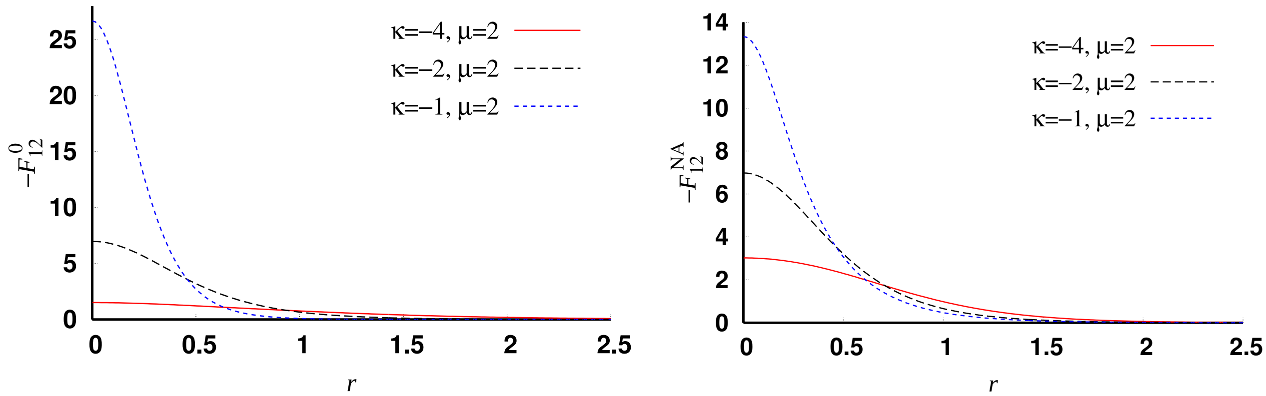
(b)

Fig. 6: (a) Profile functions for three different values of the coupling constants; a: $\kappa = -4, \mu = 2$; b: $\kappa = -2, \mu = 2$; c: $\kappa = -1, \mu = 2$; the functions are plotted in traditional style with the winding field rising linearly and the non-winding field being constant at the origin. The FI parameter $\xi = 2$. Notice that the VEV for these functions is $2^{-\frac{1}{2}}$. (b) The energy density $\mathcal{E}$ for the vortex for the same three different values of the couplings with opposite signs. All the energy densities integrate to $\pi\xi$, within an accuracy better than $\sim 10^{-4}$. Notice that the extrema of the energy density is *at the origin*, just as in the case of the ANO vortices or the non-Abelian generalizations.

winding number $k < 0$ can be interpreted as an anti-vortex. Hence, one could wonder which interpretation to give the small substructure found in this vortex solution – a small anti-vortex trapped in the non-Abelian vortex, as a bound state, not rendering the solution unstable.

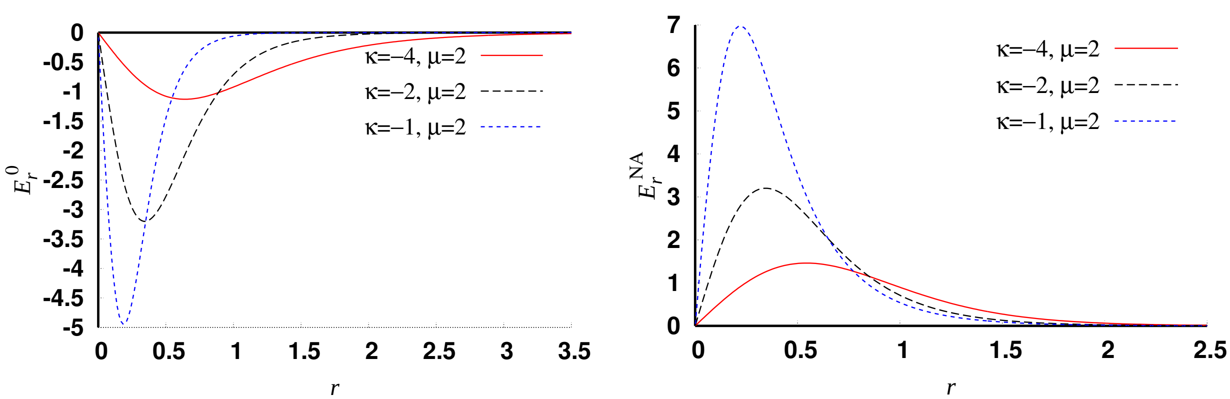
Opposite signs of coupling constants

We will now consider taking one of the couplings to be negative, say $\kappa < 0$ and $\mu > 0$. Choosing both signs negative yields the same solution as already mentioned, however with flipped electric fields. In the case of $\kappa > 0$ and $\mu < 0$, the solutions are equivalent to the ones we will consider now, just with the signs flipped of the electric fields. The Chern-Simons characteristics have been lost in this case, the vortex instead has the magnetic field concentrated at the origin – just as in the case of the ANO vortex or the single $U(N)$ non-Abelian generalization. In Fig. 6 the profile functions and energy densities for different solution are shown. In Fig. 7 the corresponding magnetic fields are shown while in Fig. 8 the electric fields are shown.



(a) (b)

Fig. 7: (a) The Abelian magnetic field F_{12}^{0} (trace-part) and (b) the non-Abelian magnetic field F_{12}^{NA} (traceless part) for three different values of the couplings with opposite signs. Notice that the magnetic field density resembles that of the ANO vortex or the non-Abelian generalizations, viz. they have the extrema at the origin. The FI parameter $\xi = 2$.



(a) (b)

Fig. 8: (a) The Abelian electric field in the radial direction E_r (trace-part) and (b) The non-Abelian electric field E_r^{NA} (traceless part) for three different values of the couplings with opposite signs. The FI parameter $\xi = 2$. Note that the electric fields are back-to-back.

4 Discussion

We have thus brought the powerful moduli matrix formalism into the non-Abelian Chern-Simons model (which supports topological non-Abelian vortices), and have conjectured that the moduli spaces of the non-Abelian vortex solutions of these systems are indeed identical to those of the vortex solutions in the Yang-Mills-Higgs models with corresponding gauge groups. We have not proved that every moduli matrix has a unique and existing solution to the master equations found. Nevertheless we have argued the plausibility of such a claim by taking the weak coupling limit which immediately yields the lumps of the Yang-Mills-Higgs models, as it is just the algebraic solutions to the D term conditions.

We have then studied some numerical solutions of non-Abelian vortices, by choosing an Ansatz to the master equations, working mainly with the $G' = SO(4)$ and $G' = USp(4)$ gauge groups. We have studied the case of different couplings with both couplings positive yielding vortex solutions with a small negative Abelian (non-Abelian) magnetic field density at the origin and a corresponding positive non-Abelian (Abelian) magnetic field density, which have a combination that is always zero (at the origin). Keeping the couplings equal provides the typical Chern-Simons characteristic that the magnetic field vanishes at the origin yielding a ring structure. These new type of solutions could perhaps be interpreted as an anti-vortex sitting inside the non-Abelian vortex as a stable bound state, with the stability provided by topological arguments.

An interesting question is to which extent this substructure found in the non-Abelian vortex solutions alters the dynamics of the vortices.

Furthermore, by changing the relative sign of the coupling constants a vortex solution with the magnetic field density concentrated at the origin has been found.

An obvious future study related to these vortices and also to the ones of Ref. [13] could be to make an explicit construction with exceptional groups and investigating the corresponding moduli spaces. Especially interesting would be the center-less groups.

Another interesting path to follow is to consider the construction of the non-Abelian vortices in Chern-Simons models with more supersymmetries, e.g. considering the model of Aharony-Bergman-Jafferis-Maldacena [39]. An Abelian non-relativistic Jackiw-Pi vortex has already been found in this model [40]. Another attempt to construct vortices in the latter model has recently been made, resulting in the non-Abelian vortex equations of the Yang-Mills-Higgs models [41].

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